Int. J. Solids Structures Vol. 23, No. 11, pp. 1505–1520, 1987
Printed in Great Britain.

0020–7683/87 $3.00+ .00

THE ENERGY IN ONE-DIMENSIONAL RATE-TYPE SEMILINEAR VISCOELASTICITY

C. FĂCIU and M. MIHĂILESCU-SULICIU
Institute of Mathematics, Str. Academiei 14, Bucharest, Romania

(*Received 6 September* 1985; *in revised form 2 December* 1986)

Abstract—The existence and properties of a free energy function compatible with the second law of thermodynamics in one-dimensional rate-type semilinear viscoelasticity is analysed. Necessary and sufficient conditions are given such that a free energy as a function of strain and stress exists and is unique, that it is a non-negative function and possesses a monotony property with respect to the equilibrium curve. A bound in energy for the smooth solutions of certain initial and boundary value problems with respect to the input data is established when the equilibrium curve is a non-monotonic curve (i.e. the free energy function is a non-convex one). Thus iron-like behaviour, for instance is also included. An L^2-approach to equilibrium is also discussed.

1. INTRODUCTION

The existence of a free energy function compatible with the second law of thermodynamics for rate-type constitutive equations has largely been investigated in the literature. References [1, 2] deal with the hypoelastic case and Refs [3, 4] with the viscoplastic and viscoelastic cases. For the elastic–perfectly plastic one-dimensional constitutive equation see Chap. 4, Section 4 in Ref. [5].

In the quasilinear viscoelastic case, the existence and uniqueness up to a constant of the free energy function is proved in Ref. [3] under the assumption that the equilibrium curve is smooth and its slope is always strictly smaller than the instantaneous slope.

For the semilinear case, if the equilibrium curve has always a strictly positive slope, it is shown in Ref. [4] that this energy function is positive, convex and possesses a monotony property with respect to the equilibrium curve. By means of the energy function one gives in Ref. [4] several energy estimates of the smooth solutions of some initial and boundary value problems.

In this paper we also consider one-dimensional rate-type semilinear viscoelastic constitutive equations. In Section 2 we give necessary and sufficient conditions, on the constitutive functions, for the free energy function to exist (and we construct it), to be non-negative and to possess a monotony property with respect to the equilibrium curve. We notice that these conditions allow the equilibrium curve to be continuous only and not necessarily increasing. These constitutive assumptions are weaker than those of Ref. [4] and more interesting since they allow us to describe materials, like iron for instance, for which the equilibrium curve has not always a positive slope (see for instance p. 577 in Ref. [6]), thus the free energy may be a non-convex function.

In Section 3 we show that all the energetic properties proved in Ref. [4] for the smooth solutions of some initial and boundary value problems still remain valid under our weaker constitutive assumptions. These properties are: the total energy at any time remains bounded by the energy of the initial data plus the energy exchanged by the body with the external world and the solution is continuously dependent upon the input data. We also obtain that in the case of an isolated body problem (i.e. for a body which does not exchange energy with the external world) there is an asymptotic L^2-approach to equilibrium. This property suggests (as in Ref. [4]) a rate-type viscoelastic approach to non-linear elasticity. But, unlike in Ref. [4], we obtain an L^2-approach of the solutions of a hyperbolic system (the viscoelastic one) to the solution of a possible non-hyperbolic system (the non-linear elastic one) since the equilibrium curve has not always a positive slope.

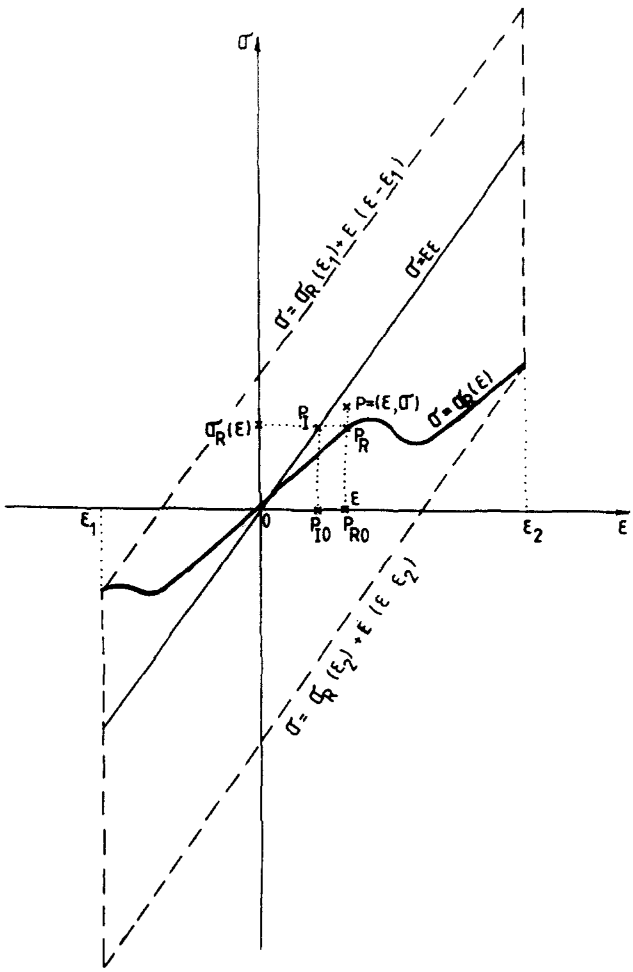

Fig. 1. The constitutive domain $\mathscr{D}$ (bounded by dotted lines) corresponding to the equilibrium curve $\sigma = \sigma_R(\varepsilon)$, $\varepsilon \in I = [\varepsilon_1, \varepsilon_2]$. For $\varepsilon \in [0, \varepsilon_2]$ the equilibrium curve is chosen here similar to that of iron in a uniaxial standard test.

We also consider the case when the constitutive domain (in the stress–strain plane) is bounded and we show that this domain cannot be arbitrarily chosen since it is determined by the equilibrium curve, and the instantaneous reponse curves (see Section 2 and Fig. 1).

2. THE FREE ENERGY FUNCTION AND SOME OF ITS PROPERTIES

Let us consider a rate-type constitutive equation (see Refs [7–9])

$$\dot{\sigma} = E\dot{\varepsilon} + G(\varepsilon, \sigma) \tag{1}$$

where $\sigma = \sigma(t)$ is the stress, $\varepsilon = \varepsilon(t)$ is the strain and

$$E = \text{const.} > 0 \tag{2a}$$

$$\begin{gathered} G: \mathscr{D} \to R, \ G \text{ Lipschitz continuous on } \mathscr{D} \\ \mathscr{D} = \{(\varepsilon, \sigma_R(\tilde{\varepsilon}) + E(\varepsilon - \tilde{\varepsilon}));\quad (\varepsilon, \tilde{\varepsilon}) \in I \times I\} \\ I \subseteq R \text{ an interval containing 0 as an interior point} \end{gathered} \tag{2b}$$

$$G = 0 \text{ if and only if } \sigma = \sigma_R(\varepsilon) \tag{2c}$$

$$(\sigma-\sigma_R(\varepsilon))G(\varepsilon,\sigma)\leqslant 0 \quad \text{for any} \quad (\varepsilon,\sigma)\in\mathscr{D} \tag{2d}$$

where

$$\sigma_R : I\to R, \quad \sigma_R(0)=0, \quad \sigma_R\in C^0(I). \tag{2e}$$

The curve $\sigma=\sigma_R(\varepsilon)$ in the ε–σ plane is called the equilibrium curve. E is the dynamic Young's modulus. Assumption (2c) ensures that the constitutive equation, eqn (1), is a viscoelastic one (unlike in Refs [7–9] where the models are viscoplastic) and condition (2a) implies that the model is semilinear (i.e. with a linear elastic instantaneous response) and has real acceleration waves.

A pair $(\varepsilon(t), \sigma(t))$, $t\in[0,T)$ is called a smooth process if $\varepsilon\in C^1([0,T))$, $\varepsilon(t)\in I$ for any $t\in[0,T)$, $(\varepsilon(0),\sigma(0))\in\mathscr{D}$ while $\sigma(t)$, $t\in[0,T)$ is the solution of eqns (1) and (2) for the given $\varepsilon(t)$ and $\sigma(0)$, such that $(\varepsilon(t),\sigma(t))\in\mathscr{D}$ for any $t\in[0,T)$. Then assumption (2d) represents the necessary and sufficient condition for the equilibrium curve to be stable with respect to relaxation processes, i.e. each constant strain process $(\varepsilon=\varepsilon_0,\ \sigma=\sigma(t))$, $t\geqslant 0$ starting at $(\varepsilon_0,\sigma(0))\in\mathscr{D}$ has infinite duration and approaches σ_R in the sense that $\lim_{t\to\infty}\sigma(t)=\sigma_R(\varepsilon_0)$. This assumption reflects some experimental evidence.

The choice of domain $\mathscr{D}$ in relation (2b) (see Fig. 1) is based on the following remark: any process starting in $\mathscr{D}$ will always remain in $\mathscr{D}$. The proof of this remark is given at the end of this section.

The constitutive equation, eqn (1), is said to have a free energy function of strain and stress, compatible with the second law of thermodynamics if there exists a smooth function $\psi=\psi(\varepsilon,\sigma)$, $\psi:\mathscr{D}\to R$ such that

$$\sigma\dot{\varepsilon}-\rho\dot{\psi}\geqslant 0$$

for any process $(\varepsilon(t),\sigma(t))$, $t\in[0,T)$ (see Ref. [1]). Here $\rho>0$ is the mass density in the reference configuration.

2.1. *Existence and uniqueness*

We give first the necessary and sufficient conditions on the constitutive functions in conditions (2) such that the constitutive equations, eqns (1) and (2), admit a free energy function. Following Ref. [4] the proof is based on the construction of this energy function.

Proposition 1. The constitutive equations, eqns (1) and (2), admit a unique free energy (modulo a constant) if and only if

$$\frac{\sigma_R(\varepsilon_1)-\sigma_R(\varepsilon_2)}{\varepsilon_1-\varepsilon_2}<E \quad \text{for any} \quad \varepsilon_1,\varepsilon_2\in I, \quad \varepsilon_1\neq\varepsilon_2. \tag{3}$$

Proof. A smooth function $\psi:\mathscr{D}\to R$ is a free energy for the constitutive equations, eqns (1) and (2), if and only if (see Ref. [3])

$$\frac{\partial\psi}{\partial\varepsilon}+E\frac{\partial\psi}{\partial\sigma}=\frac{\sigma}{\rho} \tag{4a}$$

for any $(\varepsilon,\sigma)\in\mathscr{D}$

$$\frac{\partial\psi}{\partial\sigma}G(\varepsilon,\sigma)\leqslant 0. \tag{4b}$$

Equation (4a) has the general solution

$$\rho\psi(\varepsilon,\sigma)=\frac{\sigma^2}{2E}+\phi(\sigma-E\varepsilon) \tag{5}$$

where ϕ is an arbitrary smooth function of argument $\sigma - E\varepsilon$. On the other hand, under hypothesis (2d), inequality (4b) is equivalent to

$$\frac{\partial\psi}{\partial\sigma}(\sigma-\sigma_R(\varepsilon))\geqslant 0 \quad \text{on} \quad \mathscr{D}. \tag{6}$$

Now $\partial\psi/\partial\sigma$ must vanish for $\sigma=\sigma_R(\varepsilon)$ since otherwise inequality (6) will be violated. Thus

$$\frac{\partial\psi}{\partial\sigma}(\varepsilon,\sigma_R(\varepsilon))=0 \quad \text{for any} \quad \varepsilon\in I. \tag{7}$$

By using eqn (5) in eqn (7) for $\sigma=\sigma_R(\varepsilon)$ we obtain

$$\frac{\sigma_R(\varepsilon)}{E}+\phi'(\sigma_R(\varepsilon)-E\varepsilon)=0 \quad \text{for any} \quad \varepsilon\in \mathrm{I} \tag{7'}$$

where $\phi'(\tau)\equiv \mathrm{d}\phi(\tau)/\mathrm{d}\tau$. If we denote

$$h(\varepsilon)\equiv\sigma_R(\varepsilon)-E\varepsilon \quad \text{for any} \quad \varepsilon\in I \tag{8}$$

then eqn (7′) becomes

$$\phi'(h(\varepsilon))=-\frac{\sigma_R(\varepsilon)}{E} \quad \text{for any} \quad \varepsilon\in I. \tag{9}$$

In order to determine the function $\phi(\tau)$ from eqn (9) the function $h(\varepsilon)$ has to be invertible, i.e.

$$h(\varepsilon_1)\neq h(\varepsilon_2) \quad \text{for any} \quad \varepsilon_1,\varepsilon_2\in I, \quad \varepsilon_1\neq\varepsilon_2. \tag{10}$$

When condition (10) holds, ϕ will be the unique solution (modulo a constant) of the following equation:

$$\phi'(\tau)=-\frac{\sigma_R(\bar{h}(\tau))}{E} \quad \text{for any} \quad \tau\in h(I) \tag{11}$$

where $\bar{h}$ is the inverse function of h. According to eqn (11) ϕ is defined on $h(I)$; but according to the definition of $\mathscr{D}$ in eqn (2b), for any $(\varepsilon,\sigma)\in\mathscr{D}$ there must exist an $\tilde{\varepsilon}\in I$ such that $\sigma-E\varepsilon=h(\tilde{\varepsilon})$ and therefore $\phi(\sigma-E\varepsilon)$ makes sense in eqn (5) for any $(\varepsilon,\sigma)\in\mathscr{D}$.

Now, the function ψ in eqn (5) with ϕ given by eqn (11) has also to verify inequality (6) in order to be a free energy function. Since for any $(\varepsilon,\sigma)\in\mathscr{D}$ there exists an $\tilde{\varepsilon}\in I$ such that $\sigma-E\varepsilon=h(\tilde{\varepsilon})$ we have

$$\sigma-\sigma_R(\varepsilon)=h(\tilde{\varepsilon})-h(\varepsilon)$$

and, according to eqn (5) we also have

$$\rho \frac{\partial \psi}{\partial \sigma} = \frac{\sigma}{E} + \phi'(\sigma - E\varepsilon) = \frac{E\varepsilon + h(\tilde{\varepsilon})}{E} - \frac{\sigma_R(\tilde{\varepsilon})}{E} = \varepsilon - \tilde{\varepsilon}.$$

Then inequality (6) requires

$$[h(\varepsilon) - h(\tilde{\varepsilon})](\varepsilon - \tilde{\varepsilon}) \leqslant 0 \quad \text{for any} \quad \varepsilon, \tilde{\varepsilon} \in I. \tag{12}$$

Consequently expressions (10) and (12) are the necessary and sufficient conditions for the existence of a free energy function, i.e. a solution of eqns (4a) and (4b) in $\mathscr{D}$. By using notation (8) they are equivalent to condition (3). Moreover, the solution of eqns (4a) and (4b) is unique (modulo a constant) and we can write it explicitly.

Indeed, according to expressions (10) and (12) h is strictly monotone, therefore

$$\varepsilon h(\varepsilon) = \int_0^{\varepsilon} h(s)\,\mathrm{d}s + \int_0^{h(\varepsilon)} \bar{h}(\lambda)\,\mathrm{d}\lambda \quad \text{for any} \quad \varepsilon \in I. \tag{13}$$

On the other hand from eqns (11) and (8) there results

$$\bar{h}(\tau) = -\frac{\tau}{E} - \phi'(\tau) \tag{14}$$

and by substituting eqn (14) into eqn (13) we obtain, if $\phi(0) = 0$

$$\phi(\tau) = \int_0^{\bar{h}(\tau)} \sigma_R(s)\,\mathrm{d}s - \frac{\sigma_R^2(\bar{h}(\tau))}{2E}. \tag{15}$$

The free energy function of eqns (1) and (2) for which $\psi(0, 0) = 0$ is then given by

$$\psi(\varepsilon, \sigma) = \frac{\sigma^2}{2E} + \int_0^{\bar{\varepsilon}} \sigma_R(s)\,\mathrm{d}s - \frac{\sigma_R^2(\bar{\varepsilon})}{2E} \tag{16}$$

where $\bar{\varepsilon} = \bar{h}(\sigma - E\varepsilon)$. Q.E.D.

Remark 1. In Ref. [4] the existence of a free energy function is proved under more restrictive assumptions on the equilibrium curves, i.e.

$$I = R, \quad \sigma_R \in C^1(R), \quad \sigma_R'(\varepsilon) < E \quad \text{for any} \quad \varepsilon \in R. \tag{2'}$$

From now on we assume that relation (3) holds, i.e. eqns (1) and (2) admit free energy functions. We denote by ψ that one given by expression (16) and ϕ by eqn (15).

2.2. *Properties of the free energy function*

We study now some properties of the free energy function ψ which are important from both the physical and mathematical point of view. The first property we investigate is the non-negativeness of ψ. In general the free energy is not required to be a non-negative function; however, in many cases, it is supposed to be non-negative so it is interesting to know under what conditions such a property holds. Moreover, a non-negative function ϕ is very useful in studying the solutions of the system of equations which describes the motion of a body modeled by the constitutive equations, eqns (1) and (2) (see Section 3).

Proposition 2. The function ϕ is non-negative on $h(I)$ if and only if

$$\int_0^{\varepsilon} \sigma_R(s)\, \mathrm{d}s \geqslant \frac{\sigma_R^2(\varepsilon)}{2E} \quad \text{for any} \quad \varepsilon \in I. \tag{17}$$

If $I = R$, relation (17) is equivalent to

$$\int_0^{\varepsilon} \sigma_R(s)\, \mathrm{d}s \geqslant 0 \quad \text{for any} \quad \varepsilon \in R. \tag{17'}$$

Proof. According to eqn (15) relation (17) is obviously equivalent to $\phi(\tau) \geqslant 0$ for any $\tau \in h(I)$.

Let us prove now that for $I = R$ relation (17) is equivalent to relation (17′). We have only to prove that relation (17′) implies relation (17) since the converse is obvious. Assume then that relation (17′) holds and choose an $\varepsilon^* \in R$. We denote

$$\bar{\varepsilon} = \varepsilon^* - \frac{\sigma_R(\varepsilon^*)}{E}.$$

Then

$$\int_0^{\varepsilon^*} \sigma_R(s)\, \mathrm{d}s = \int_0^{\bar{\varepsilon}} \sigma_R(s)\, \mathrm{d}s + \int_{\bar{\varepsilon}}^{\varepsilon^*} \sigma_R(s)\, \mathrm{d}s \geqslant \int_{\bar{\varepsilon}}^{\varepsilon^*} \sigma_R(s)\, \mathrm{d}s.$$

If $\bar{\varepsilon} < \varepsilon^*$ one has, according to inequality (3)

$$\sigma_R(s) - \sigma_R(\varepsilon^*) > E(s - \varepsilon^*) \quad \text{for any} \quad s \in [\bar{\varepsilon}, \varepsilon^*)$$

which implies

$$\int_{\bar{\varepsilon}}^{\varepsilon^*} \sigma_R(s)\, \mathrm{d}s > [\sigma_R(\varepsilon^*) - E\varepsilon^*](\varepsilon^* - \bar{\varepsilon}) + \frac{E}{2}(\varepsilon^{*2} - \bar{\varepsilon}^2) = \frac{\sigma_R^2(\varepsilon^*)}{2E}$$

and therefore relation (17) holds for ε^*.

If $\bar{\varepsilon} > \varepsilon^*$ the proof follows in the same way. Q.E.D.

Remark 2. In general, when $I \neq R$, relations (17) and (17′) are no more equivalent. If, for instance, $I = [\varepsilon_1, \varepsilon_2]$ and $\sigma = \sigma_R(\varepsilon)$ is such that $\sigma_R(\varepsilon_2) < 0$ and $\int_0^{\varepsilon_2} \sigma_R(s)\, \mathrm{d}s = 0$ then relation (17′) holds while relation (17) is obviously violated for $\varepsilon = \varepsilon_2$.

In particular, if

$$I = [\varepsilon_1, \varepsilon_2], \quad \sigma_R(\varepsilon_1) \leqslant 0, \quad \sigma_R(\varepsilon_2) \geqslant 0 \tag{18}$$

then relations (17) and (17′) are equivalent on I.

Corollary 1. When $I = R$ or $I \neq R$ but condition (18) holds then relation (17′) is the necessary and sufficient condition for the free energy function ψ to be non-negative. When $I \neq R$ relation (17) is a sufficient condition for the free energy function ψ to be non-negative.

Remark 3. In Ref. [4] the free energy on $R^2 - \{0\}$ is proved to be positive under the assumption that the equilibrium curve is always a strictly increasing function on R and satisfies condition (2′), i.e.

$$0 < \sigma_R'(\varepsilon) < E \quad \text{for any} \quad \varepsilon \in R.$$

This condition obviously implies relation (17′).

It is interesting to note that in Ref. [4] the slope of the equilibrium curve is required to be strictly positive while relation (17′) requires that only the area defined by the equilibrium curve be positive for any $\varepsilon \in R$.

We give now a necessary and sufficient condition for ψ to possess a monotony property with respect to the equilibrium curve.

Proposition 3. If we have two functions $G_i(\varepsilon, \sigma)$, $i = 1, 2$ and two equilibrium curves $\sigma_{R_i}(\varepsilon)$, $i = 1, 2$ such that each pair (σ_{R_i}, G_i), $i = 1, 2$ satisfies conditions (2) and (3) for $I = R$ then

$$\psi_1(\varepsilon, \sigma) \geqslant \psi_2(\varepsilon, \sigma) \quad \text{for any} \quad (\varepsilon, \sigma) \in R^2 \tag{19}$$

if and only if

$$\int_0^\varepsilon \sigma_{R_1}(s)\, \mathrm{d}s \geqslant \int_0^\varepsilon \sigma_{R_2}(s)\, \mathrm{d}s \quad \text{for any} \quad \varepsilon \in R \tag{20}$$

where ψ_i is the free energy function (16) corresponding to the pair (σ_{R_i}, G_i), $i = 1, 2$.

Proof. Let τ be such that

$$\tau = h_1(\varepsilon_1) = h_2(\varepsilon_2), \quad h_i(\varepsilon) \equiv \sigma_{R_i}(\varepsilon) - E\varepsilon, \quad i = 1, 2, \quad \varepsilon \in R. \tag{21}$$

Then, according to expressions (15) and (21)

$$\begin{aligned} \phi_1(\tau) - \phi_2(\tau) &= \int_0^{\varepsilon_1} \sigma_{R_1}(s)\, \mathrm{d}s - \int_0^{\varepsilon_2} \sigma_{R_2}(s)\, \mathrm{d}s - \frac{1}{2E}[\sigma_{R_1}^2(\varepsilon_1) - \sigma_{R_2}^2(\varepsilon_2)] \\ \sigma_{R_1}(\varepsilon_1) - \sigma_{R_2}(\varepsilon_2) &= E(\varepsilon_1 - \varepsilon_2). \end{aligned} \tag{22}$$

By a simple calculation we get from expressions (22)

$$\begin{aligned} \phi_1(\tau) - \phi_2(\tau) &= \int_0^{\varepsilon_1} [\sigma_{R_1}(s) - \sigma_{R_2}(s)]\, \mathrm{d}s + \int_{\varepsilon_2}^{\varepsilon_1} \sigma_{R_2}(s)\, \mathrm{d}s - (\varepsilon_1 - \varepsilon_2)\sigma_{R_2}(\varepsilon_2) - \frac{E}{2}(\varepsilon_1 - \varepsilon_2)^2 \\ &= \int_0^{\varepsilon_1} [\sigma_{R_1}(s) - \sigma_{R_2}(s)]\, \mathrm{d}s + \int_{\varepsilon_2}^{\varepsilon_1} [\sigma_{R_2}(s) - \sigma_{R_2}(\varepsilon_2) - E(s - \varepsilon_2)]\, \mathrm{d}s. \end{aligned} \tag{23}$$

A similar calculation leads to

$$\phi_1(\tau) - \phi_2(\tau) = \int_0^{\varepsilon_2} [\sigma_{R_1}(s) - \sigma_{R_2}(s)]\, \mathrm{d}s + \int_{\varepsilon_2}^{\varepsilon_1} [\sigma_{R_1}(s) - \sigma_{R_1}(\varepsilon_1) - E(s - \varepsilon_1)]\, \mathrm{d}s. \tag{24}$$

But, according to condition (3) we have

$$\left.\begin{aligned} &\int_{\varepsilon_2}^{\varepsilon_1} [\sigma_{R_2}(s) - \sigma_{R_2}(\varepsilon_2) - E(s - \varepsilon_2)]\, \mathrm{d}s < 0 \\ &\int_{\varepsilon_2}^{\varepsilon_1} [\sigma_{R_1}(s) - \sigma_{R_1}(\varepsilon_1) - E(s - \varepsilon_1)]\, \mathrm{d}s > 0 \end{aligned}\right\} \quad \text{for any} \quad \varepsilon_1, \varepsilon_2 \in R, \varepsilon_1 \neq \varepsilon_2$$

and therefore, from relations (23) and (24) we obtain

$$\int_0^{\varepsilon_2} [\sigma_{R_1}(s)-\sigma_{R_2}(s)]\,\mathrm{d}s < \phi_1(\tau)-\phi_2(\tau) < \int_0^{\varepsilon_1} [\sigma_{R_1}(s)-\sigma_{R_2}(s)]\,\mathrm{d}s. \tag{25}$$

Thus $\phi_1(\tau) \geqslant \phi_2(\tau)$ for any $\tau \in R$ if and only if inequality (20) holds, which proves the equivalence of inequalities (19) and (20). Q.E.D.

Remark 4. The monotony property of the free energy function ψ for (1)+(2)+(2′) is proved in Ref. [4] when

$$0 < \sigma'_{R_2}(\varepsilon) \leqslant \sigma'_{R_1}(\varepsilon) < E \quad \text{for any} \quad \varepsilon \in R.$$

This case is obviously included in the hypotheses of Proposition 3.

A possible physical interpretation of the monotony property is given in Ref [4]: if a viscoelastic material characterized by a constitutive equation, eqns (1) and (2), is subject to a certain thermal process (such as annealing or quenching for instance) that leaves the Young's modulus E unchanged but changes the function G and the equilibrium curve according to inequality (20) then we are able to compare the free energies for the two constitutive equations.

The monotony property may also be used when comparing the free energy ψ to an Euclidean norm on R^2 (see Section 3).

The constitutive domain $\mathscr{D}$. We will now justify the choice of the constitutive domain $\mathscr{D}$ in (2b). We prove first that any process $(\varepsilon(t), \sigma(t))$, $t \in [0, T)$ starting in $\mathscr{D}$, i.e. $(\varepsilon(0), \sigma(0)) \in \mathscr{D}$, will remain in $\mathscr{D}$ for $t \in [0, T)$. Indeed, let us denote

$$p(t) = \sigma(t) - E\varepsilon(t).$$

Then

$$\dot{p}(t) = G(\varepsilon(t), \sigma(t)).$$

Now, if $I = (a, b)$ then $h(I) = (\alpha, \beta)$; for $(\varepsilon(0), \sigma(0)) \in \mathscr{D}$ there always exists a $T > 0$ such that $p(t) \in h(I)$ for $t \in [0, T)$. Suppose that T is such that $p(T) = \beta$, $p(t) \in h(I)$ for $t \in [0, T)$. Then

$$p(T) = (\sigma - \sigma_R(\varepsilon))(T) + h(\varepsilon(T)) = \beta$$

thus

$$(\sigma - \sigma_R(\varepsilon))(T) = \beta - h(\varepsilon(T)) > 0$$

and therefore according to (2d) $G(\varepsilon(T), \sigma(T)) < 0$. Then

$$\dot{p}(T) = G(\varepsilon(T), \sigma(T)) < 0$$

and there results $p(T-\delta) > p(T) = \beta$ for $\delta > 0$ sufficiently small which contradicts the hypothesis $p(t) < \beta$ for $t \in [0, T)$.

If $I = [a, b]$ then $h(I) = [h(b), h(a)]$ and for any t for which $p(t) \geqslant h(a)$ we have $(\sigma - \sigma_R(\varepsilon))(t) = p(t) - h(\varepsilon(t)) \geqslant h(a) - h(\varepsilon(t)) \geqslant 0$ which, according to (2d), implies

$$\dot{p}(t) = G(\varepsilon(t), \quad \sigma(t)) \leqslant 0.$$

Therefore, $p(t) \leqslant h(a)$ for any $t \geqslant 0$ as $p(0) \leqslant h(a)$.

In conclusion, if G is defined on a domain $\mathscr{D}^* \supset \mathscr{D}$ the states in $\mathscr{D}^*$ which are not in $\mathscr{D}$ cannot be reached by processes starting at states in $\mathscr{D}$. Thus, from the mathematical point of view there is no need to define G outside $\mathscr{D}$; from the practical point of view we are not able to determine G experimentally outside $\mathscr{D}$ since the states outside $\mathscr{D}$ cannot be reached by processes starting in $\mathscr{D}$.

Now, it seems quite natural for any state (ε, σ) reached by a process to possess a free energy function and, according to expressions (15) and (16) the domain $\mathscr{D}$ is precisely the set of all states (ε, σ), $\varepsilon \in I$ for which we could prove there exists a free energy function.

3. SOME ENERGY ESTIMATES IN ONE-DIMENSIONAL VISCOELASTICITY

Let us consider the system of partial differential equations describing the motion of a rate-type semilinear viscoelastic material

$$\begin{aligned} \rho\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} &= \rho b \\ \frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} &= 0 \\ \frac{\partial \sigma}{\partial t} - E\frac{\partial \varepsilon}{\partial t} &= G(\varepsilon, \sigma). \end{aligned} \tag{26}$$

Here $\varepsilon(x, t)$, $\sigma(x, t)$, $v(x, t)$ are the strain, stress and particle velocity, respectively, $\rho > 0$ is the initial mass density, $b(x, t)$ is the body force and to the constitutive equation, eqn $(26)_3$, we add the constitutive assumptions (2), (3) and (17) (or (17′) if $I = R$) such that eqn $(26)_3$ admits a unique positive free energy function ψ with $\psi(0,0) = 0$.

We consider the following problem for system (26)

$$\begin{aligned} (\varepsilon, \sigma, v)(x, 0) &= (\varepsilon_0, \sigma_0, v_0)(x), \quad x \in (0, l) \\ (\sigma v)(0, t) &= (\sigma v)(l, t) = 0, \quad t > 0 \end{aligned} \tag{27}$$

where the initial data are such as not to generate acceleration or shock waves at $t = 0+$.

In the following by $(\varepsilon, \sigma, v)(x, t)$ we will always denote a smooth solution of problem (26) and (27).

Now let $\tilde{\psi}$ be any smooth solution of the energy equation, eqn (4a). We define, the energy $\tilde{e}_{\tilde{\psi}}$ of the solution $(\varepsilon, \sigma, v)(x, t)$ of problem (26) and (27) by

$$\tilde{e}_{\tilde{\psi}}(t) \equiv \tilde{e}_{\tilde{\psi}}(\varepsilon(.,t), \sigma(.,t), v(.,t)) = \int_0^l \left[\frac{\rho}{2}v^2 + \rho\tilde{\psi}(\varepsilon, \sigma)\right](x, t)\,\mathrm{d}x, \quad t \geqslant 0. \tag{28}$$

Then the following energy identity holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{e}_{\tilde{\psi}}(t) = \int_0^l \left[\rho v b + \rho\frac{\partial\tilde{\psi}}{\partial\sigma}G\right]\mathrm{d}x, \quad t \geqslant 0. \tag{29}$$

Indeed, from eqns (26) and (4a) we have

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}v^2\right) = v\frac{\partial\sigma}{\partial x} + \rho vb = \frac{\partial}{\partial x}(\sigma v) - \sigma\frac{\partial\varepsilon}{\partial t} + \rho vb$$

$$= \frac{\partial}{\partial x}(\sigma v) + \rho vb - \frac{\partial\varepsilon}{\partial t}\left(\rho\frac{\partial\tilde{\psi}}{\partial\varepsilon} + \rho E\frac{\partial\tilde{\psi}}{\partial\sigma}\right)$$

$$= \frac{\partial}{\partial x}(\sigma v) + \rho vb - \rho\frac{\partial\tilde{\psi}}{\partial t} + \rho\frac{\partial\tilde{\psi}}{\partial\sigma}G.$$

Relation (29) follows now immediately if we integrate with respect to x and take condition $(27)_2$ into account.

Function (28) corresponding to the free energy function ψ given by expression (16) will be denoted by $e(t)$ and will be called the *total energy* of the solution $(\varepsilon, \sigma, v)(x, t)$. We recall here that the free energy ψ is that solution of eqn (4a) which also verifies (4b) and $\psi(0,0) = 0$.

The energies $\tilde{e}_{\tilde{\psi}}(t)$ defined by expression (28) have been introduced in Ref. [4] where the total energy $e(t)$, under assumptions (2′) and $\sigma'_R(\varepsilon) > 0$ for any $\varepsilon \in R$, is used to establish several important properties of the smooth solutions of problem (26) and (27). In the following we prove that some of these properties are still valid when the assumptions of Ref. [4] are weakened in the sense that they are replaced by relations (3) and (17) (or (17′) if $I = R$).

3.1. *Bounds in energy and stability*

We start with a result which shows how the total energy $e(t)$ may be used to obtain a bound in L^2 for the solution $(\varepsilon, \sigma, v)(x, t)$ of problem (26) and (27).

Proposition 4. If $e(0) < \infty$ and if there exists $E_1 = \text{const.}$, $0 < E_1 < E$, such that

$$\int_0^\varepsilon \sigma_R(s)\,\mathrm{d}s \geqslant \frac{E_1\varepsilon^2}{2} + \frac{1}{2(E-E_1)}[\sigma_R(\varepsilon) - E_1\varepsilon]^2 \quad \text{for any} \quad \varepsilon \in I \tag{30}$$

then

$$\left(\int_0^l \frac{E}{2}\varepsilon^2(x,t)\,\mathrm{d}x\right)^{1/2}, \left(\int_0^l \frac{1}{2E}\sigma^2(x,t)\,\mathrm{d}x\right)^{1/2}, \left(\int_0^l \frac{\rho}{2}v^2(x,t)\,\mathrm{d}x\right)^{1/2}$$

$$\leqslant K\left[\sqrt{(e(0))} + \frac{1}{\sqrt{2}}\int_0^t\left(\int_0^l \rho b^2(x,s)\,\mathrm{d}x\right)^{1/2}\mathrm{d}s\right] \quad \text{for any} \quad t \geqslant 0 \tag{31}$$

where $K = \sqrt{(2E/E_1)}$.

Proof. Consider the function ϕ given by expression (15). We prove first that

$$\phi(\tau) \geqslant A\tau^2, \quad A = \text{const.} > 0 \quad \text{for any} \quad \tau \in h(I) \tag{32}$$

if and only if there exists $E_1 = \text{const.}$ with $0 < E_1 < E$ such that inequality (30) holds and then

$$E_1 = \frac{2AE^2}{1+2AE}.$$

Indeed, this statement follows immediately from expression (15) since

$$\phi(h(\varepsilon)) - Ah^2(\varepsilon) = \int_0^\varepsilon\left[\sigma_R(s) - \frac{2AE^2}{1+2AE}s\right]\mathrm{d}s - \left(A + \frac{1}{2E}\right)\left[\sigma_R(\varepsilon) - \frac{2AE^2}{1+2AE}\varepsilon\right]^2.$$

Now, as $(\partial\psi/\partial\sigma)G \leqslant 0$, there results from eqn (29) (written for the total energy $e(t)$)

$$\dot{e}(t) \leqslant \left(\int_0^l \frac{\rho}{2} v^2 \, dx\right)^{1/2} \left(\int_0^l 2\rho b^2 \, dx\right)^{1/2} \leqslant \sqrt{(e(t))} \left(\int_0^l 2\rho b^2 \, dx\right)^{1/2}.$$

Thus

$$\sqrt{(e(t))} \leqslant \sqrt{(e(0))} + \frac{1}{\sqrt{2}} \int_0^t \left(\int_0^l \rho b^2(x,s) \, dx\right)^{1/2} ds \quad \text{for any} \quad t \geqslant 0. \tag{33}$$

Therefore, the square root of the total energy at any time $t \geqslant 0$ is bounded from above by the square root of the total energy of the initial data plus the energy supplied to the body by the external world during the whole time interval $[0, t]$.

Now, from relations (33) and (28) there results

$$\left[\int_0^l \frac{\sigma^2}{2E}(x,t) \, dx\right]^{1/2}, \left[\int_0^l \frac{\rho}{2} v^2(x,t) \, dx\right]^{1/2},$$
$$\left[\int_0^l \phi((\sigma - E\varepsilon)(x,t)) \, dx\right]^{1/2} \leqslant \sqrt{(e(0))} + \frac{1}{\sqrt{2}} \int_0^t \left(\int_0^l \rho b^2(x,s) \, dx\right)^{1/2} ds \equiv M(t).$$

But, according to the above remark, inequalities (30) and (32) are equivalent and

$$A = \frac{E_1}{2E(E-E_1)}.$$

Thus we have

$$\int_0^l \frac{E}{2} \varepsilon^2(x,t) \, dx \leqslant \int_0^l \frac{1}{E} \sigma^2(x,t) \, dx + \int_0^l \frac{1}{E} (\sigma - E\varepsilon)^2(x,t) \, dx \leqslant \left(2 + \frac{1}{AE}\right) M^2(t) = \frac{2E}{E_1} M^2(t)$$

and (31) follows. Q.E.D.

Remark 5. If $I = R$ condition (30) is equivalent to

$$\int_0^\varepsilon \sigma_R(s) \, ds \geqslant \frac{E_1 \varepsilon^2}{2} \quad \text{for any} \quad \varepsilon \in R.$$

The proof is immediate if we use relation (17′).

This equivalence is also a direct consequence of the monotony property given by Proposition 3 when $\sigma_{R_1}(\varepsilon) = \sigma_R(\varepsilon)$, $\sigma_{R_2}(\varepsilon) = E_1 \varepsilon$ for any $\varepsilon \in R$ and $\phi_1(\tau) = \phi(\tau)$, $\phi_2(\tau) = E_1 \tau^2 / [2E(E - E_1)]$ for any $\tau \in R$.

The upper bound in L^2 given by (31), but stated in slightly different terms is obtained in Ref. [4] under the assumption that there exists an $E_1 = \text{const.}$, $0 < E_1 < E$ with

$$0 < E_1 \leqslant \sigma_R'(\varepsilon) \quad \text{for any} \quad \varepsilon \in R. \tag{30′}$$

Condition (30′) obviously implies inequality (30) for a smooth σ_R but inequality (30) is less restrictive than condition (30′) even in the case when σ_R is smooth (for instance σ_R is no longer required to be an increasing function on R).

We end this subsection by some remarks concerning the continuous dependence of the solutions of problem (26) and (27) upon the input data. By input data we understand the initial data $(\varepsilon_0, \sigma_0, v_0)(x)$ and the body force $b(x,t)$.

Let $(\varepsilon_0^i, \sigma_0^i, v_0^i)(x)$, $b^i(x,t)$, $i = 1,2$ be two sets of input data and let $(\varepsilon^i, \sigma^i, v^i)(x,t)$, $i = 1,2$ be the corresponding solutions of problem (26) and (27). We denote

$$(\varepsilon, \sigma, v)(x,t) = (\varepsilon^1 - \varepsilon^2, \sigma^1 - \sigma^2, v^1 - v^2)(x,t)$$
$$(\varepsilon_0, \sigma_0, v_0)(x) = (\varepsilon_0^1 - \varepsilon_0^2, \sigma_0^1 - \sigma_0^2, v_0^1 - v_0^2)(x)$$
$$b(x,t) = (b^1 - b^2)(x,t)$$

and require that $(\sigma v)(0,t) = (\sigma v)(l,t) = 0$, $t > 0$.

Let us also denote

$$N(t) \equiv N(\varepsilon(.,t), \sigma(.,t), v(.,t)) = \int_0^l \left(\frac{\rho}{2}v^2 + \frac{\sigma^2}{2E} + \frac{E}{2}\varepsilon^2\right)(x,t)\,\mathrm{d}x \quad \text{for any} \quad t \geqslant 0.$$

We give the following known result from the theory of stability for partial differential equations (see Chap. II, Section 11 in Ref. [10] for instance).

There exists a constant $K > 0$ independent of the input data such that

$$\sqrt{(N(t))} \leqslant \left\{\sqrt{(N(0))} + \int_0^t \left(\int_0^l \frac{\rho}{2} b^2\,\mathrm{d}x\right)^{1/2} \exp(-Ks)\,\mathrm{d}s\right\} \exp(Kt) \quad \text{for any} \quad t \geqslant 0. \tag{34}$$

Relation (34) implies that the solution of problem (26) and (27) is unique and is continuously dependent on the input data, with respect to the norm

$$\|(\varepsilon, \sigma, v)(.,.)\| = \sup_{t \geqslant 0} \sqrt{(N(t))}.$$

Now suppose the total energy $e(t)$ is "equivalent" to $N(t)$ in the following sense: there exist two positive constants A_1, A_2, $A_1 < A_2$, such that

$$A_1 N(t) \leqslant e(t) \leqslant A_2 N(t) \quad \text{for any} \quad t \geqslant 0 \tag{35}$$

for any given function $(\varepsilon, \sigma, v)(x,t)$ for which $e(t)$ and $N(t)$ make sense. Then obviously any smooth solution of problem (26) and (27) will be continuously dependent on the input data, with respect to the total energy.

Proposition 5. Relation (35) holds if and only if there exist two positive constants E_1, E_2, $E_1 < E_2 < E$ such that, for any $\varepsilon \in I$

$$\frac{E_1\varepsilon^2}{2} + \frac{1}{2(E-E_1)}[\sigma_R(\varepsilon) - E_1\varepsilon]^2 \leqslant \int_0^\varepsilon \sigma_R(s)\,\mathrm{d}s \leqslant \frac{E_2\varepsilon^2}{2} + \frac{1}{2(E-E_2)}[\sigma_R(\varepsilon) - E_2\varepsilon]^2. \tag{36}$$

Then $A_1 = E_1/[2E(E-E_1)]$, $A_2 = E_2/[2E(E-E_2)]$.

If $I = R$ condition (36) is equivalent to

$$\tfrac{1}{2}E_1\varepsilon^2 \leqslant \int_0^\varepsilon \sigma_R(s)\,\mathrm{d}s \leqslant \tfrac{1}{2}E_2\varepsilon^2 \quad \text{for any} \quad \varepsilon \in R. \tag{37}$$

The proof is simple if we take into account the equivalence of inequalities (30) and (32) proved in Proposition 4.

3.2. *Approach to equilibrium*

Let us consider now the case when

$$\begin{aligned} G(\varepsilon,\sigma) &= -k_0(\sigma-\sigma_R(\varepsilon)) \quad \text{for any} \quad (\varepsilon,\sigma)\in\mathscr{D} \\ k_0 &= \text{const.} > 0. \end{aligned} \tag{38}$$

For this particular form of the function G it is proved in Ref. [4] that, if

$$I = R, \quad \sigma_R \in C^1(R) \quad \text{and} \quad 0 < \sigma_R'(\varepsilon) < E \quad \text{for any} \quad \varepsilon \in R$$

then

$$\int_0^t \int_0^l [\sigma(x,s)-\sigma_R(\varepsilon(x,s))]^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{Ee(0)}{k_0}, \quad t>0 \tag{39}$$

for the solution $(\varepsilon,\sigma,v)(x,t)$ of an isolated body problem, i.e. problem (26) and (27) with $b=0$. Therefore, when $k_0\to\infty$, relation (39) implies an L^2-approach to equilibrium for the solution of an isolated body problem.

We show here that a similar result may be obtained even when σ_R is no longer an increasing function. Such a result may be useful because system (26) is always hyperbolic while the "equilibrium system" given by eqns $(26)_{1,2}$ and the constitutive equation $\sigma=\sigma_R(\varepsilon)$ (i.e. the non-linear elastic system) may loose its hyperbolic character in this case. However, we still obtain an L^2-approach of the solutions of a hyperbolic system to the solution of a non-hyperbolic system.

Proposition 6. Let G be given by expression (38) and let $b=0$. If there exists a constant $B>0$ such that

$$E-\frac{1}{B} \leqslant \frac{\sigma_R(\varepsilon_1)-\sigma_R(\varepsilon_2)}{\varepsilon_1-\varepsilon_2} \quad \text{for any} \quad \varepsilon_1,\varepsilon_2\in I \quad \varepsilon_1\neq\varepsilon_2 \tag{40}$$

then

$$\int_0^t \int_0^l [\sigma(x,s)-\sigma_R(\varepsilon(x,s))]^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{e(0)}{k_0 B} \quad \text{for any} \quad t>0. \tag{41}$$

Proof. We prove first that inequality (40) is equivalent to

$$\rho\frac{\partial\psi}{\partial\sigma}(\sigma-\sigma_R(\varepsilon)) \geqslant B(\sigma-\sigma_R(\varepsilon))^2 \quad \text{for any} \quad (\varepsilon,\sigma)\in\mathscr{D}. \tag{42}$$

Indeed, there exists an $\tilde{\varepsilon}\in I$ such that $\sigma-E\varepsilon=h(\tilde{\varepsilon})$; then (see the proof to Proposition 1)

$$\rho\frac{\partial\psi}{\partial\sigma}=\varepsilon-\tilde{\varepsilon}$$
$$\sigma-\sigma_R(\varepsilon)=h(\tilde{\varepsilon})-h(\varepsilon)=E(\varepsilon-\tilde{\varepsilon})-[\sigma_R(\varepsilon)-\sigma_R(\tilde{\varepsilon})] \tag{43}$$

and therefore

$$\rho E\frac{\partial\psi}{\partial\sigma}=\sigma-\sigma_R(\varepsilon)+[\sigma_R(\varepsilon)-\sigma_R(\tilde{\varepsilon})].$$

Let us denote

$$A\equiv\rho\frac{\partial\psi}{\partial\sigma}(\sigma-\sigma_R(\varepsilon))-B(\sigma-\sigma_R(\varepsilon))^2.$$

We have

$$A\left(\rho\frac{\partial\psi}{\partial\sigma}\right)^2=\rho\frac{\partial\psi}{\partial\sigma}(\sigma-\sigma_R(\varepsilon))\left[\left(\rho\frac{\partial\psi}{\partial\sigma}\right)^2-B\rho\frac{\partial\psi}{\partial\sigma}(\sigma-\sigma_R(\varepsilon))\right].$$

But, by using relations (43)

$$\left(\rho\frac{\partial\psi}{\partial\sigma}\right)^2-B\rho\frac{\partial\psi}{\partial\sigma}(\sigma-\sigma_R(\varepsilon))=(1-BE)(\varepsilon-\tilde{\varepsilon})^2+B[\sigma_R(\varepsilon)-\sigma_R(\tilde{\varepsilon})](\varepsilon-\tilde{\varepsilon})$$

thus

$$A\left(\rho\frac{\partial\psi}{\partial\sigma}\right)^2=\rho\frac{\partial\psi}{\partial\sigma}[\sigma-\sigma_R(\varepsilon)][(1-BE)(\varepsilon-\tilde{\varepsilon})^2+B(\sigma_R(\varepsilon)-\sigma_R(\tilde{\varepsilon}))(\varepsilon-\tilde{\varepsilon})]. \tag{44}$$

Relation (44) proves the equivalence of inequalities (40) and (42) since $\partial\psi/\partial\sigma$ and $\sigma-\sigma_R(\varepsilon)$ have always the same sign and vanish simultaneously (see relation (6)).

Now, by using (42) we may write (29) (for $\tilde{e}_{\tilde{\psi}}=e$)

$$\dot{e}(t)=-k_0\int_0^l\rho\frac{\partial\psi}{\partial\sigma}(\sigma-\sigma_R(\varepsilon))(x,t)\,\mathrm{d}x\leqslant-k_0B\int_0^l[\sigma-\sigma_R(\varepsilon)]^2(x,t)\,\mathrm{d}x$$

and therefore

$$\int_0^t\int_0^l[\sigma(x,s)-\sigma_R(\varepsilon(x,s))]^2\,\mathrm{d}x\,\mathrm{d}s\leqslant-\frac{1}{k_0B}[e(t)-e(0)]\leqslant\frac{e(0)}{k_0B}. \qquad \text{Q.E.D.}$$

Remark 6. If we consider

$$G(\varepsilon,\sigma)=-k(\varepsilon,\sigma)(\sigma-\sigma_R(\varepsilon))\quad\text{for any}\quad(\varepsilon,\sigma)\in\mathscr{D}$$

with

$$k(\varepsilon,\sigma)>k_0=\text{const.}>0\qquad\text{for any}\quad(\varepsilon,\sigma)\in\mathscr{D}$$

Proposition 6 still holds. The proof is essentially the same.

3.3. *Additional comments and concluding remarks*

Let us introduce the equilibrium energy function of the viscoelastic model, eqns (1) and (2), defined by (see expression (16) where $\bar{\varepsilon} = \bar{h}(\sigma_R(\varepsilon) - E\varepsilon) = \bar{h}(h(\varepsilon)) = \varepsilon$)

$$\psi_R(\varepsilon) \equiv \psi(\varepsilon, \sigma_R(\varepsilon)) = \int_0^\varepsilon \sigma_R(s)\, \mathrm{d}s \quad \text{for any} \quad \varepsilon \in I \tag{45}$$

and the instantaneous energy function defined by

$$\psi_I(\varepsilon) = \tfrac{1}{2}E\varepsilon^2 \quad \text{for any} \quad \varepsilon \in I. \tag{46}$$

We note that $\psi_R(\varepsilon)$ is exactly the strain-energy function corresponding to the non-linear elastic constitutive equation $\sigma = \sigma_R(\varepsilon)$ while $\psi_I(\varepsilon)$ is the strain-energy function corresponding to the linear elastic constitutive equation $\sigma = E\varepsilon$.

In terms of the free energy function $\psi(\varepsilon, \sigma)$ (given by expression (16)), the equilibrium energy function $\psi_R(\varepsilon)$ and of the instantaneous energy function $\psi_I(\varepsilon)$, most of the results obtained in Sections 2 and 3 can also be described as follows.

(a) For a continuous equilibrium curve $\sigma = \sigma_R(\varepsilon)$ there is a unique free energy function $\psi(\varepsilon, \sigma)$ of the constitutive equation, eqns (1) and (2), if and only if the slope of the straight line connecting any two points on the equilibrium curve is bounded from above by instantaneous Young's modulus E (Proposition 1, condition (3)). If the slope of the straight line connecting any two points on the equilibrium curve is also bounded from below by a (not necessarily positive) constant we obtain a viscoelastic approach (in L^2 sense) to non-linear elasticity (in case of an isolated body problem) (Proposition 6, condition (40)).

(b) The free energy function is non-negative at any point $P = (\varepsilon, \sigma) \in \mathscr{D}$ if the instantaneous energy function at the strain $\sigma_R(\varepsilon)/E$ (the area $0P_IP_{I0}$ in Fig. 1) does not exceed the equilibrium energy function at the strain ε (the area $0P_RP_{R0}$ in Fig. 1) (Proposition 2, condition (17)). When the equilibrium curve is defined on the whole real line we have $\psi(\varepsilon, \sigma) \geqslant 0$ on $\mathscr{D}$ if and only if $\psi_R(\varepsilon) \geqslant 0$ on R (Proposition 2, condition (17')).

(c) The monotony property (Proposition 3) states that the free energy functions for two viscoelastic models of type (1) and (2) with the same dynamic Young's modulus E are ordered if and only if the corresponding equilibrium energy functions are ordered in the same way.

(d) In the case when the equilibrium curve is defined on the whole real line, the solutions of the initial boundary value problem (26) and (27) are bounded (in L^2) if there exists a positive constant $E_1 < E$ such that the equilibrium energy function is bounded from below by the strain-energy function of the linear constitutive equation $\sigma = E_1\varepsilon$ (Proposition 4 and Remark 5). If the equilibrium energy function is also bounded from above by the strain-energy function of a linear constitutive equation $\sigma = E_2\varepsilon$ where $0 < E_1 < E_2 < E$ then the solutions of problem (26) and (27) are continuously dependent on the input data (Proposition 5).

(e) All the above conditions do not require the equilibrium function $\sigma_R(\varepsilon)$ to be an increasing function and therefore the free energy function $\psi(\varepsilon, \sigma)$ may be nonconvex.

REFERENCES

1. M. Mihăilescu-Suliciu and I. Suliciu, Energy for hypoelastic constitutive equations. *Archs Ration. Mech. Analysis* **70,** 168–179 (1979).
2. B. D. Coleman and D. R. Owen, On thermodynamics and intrinsically equilibrated materials. *Annali Mat. Pura Appl.* (IV) **108,** 189–199 (1976).
3. M. E. Gurtin, W. O. Williams and I. Suliciu, On rate type constitutive equations and the energy of viscoelastic and viscoplastic materials. *Int. J. Solids Structures* **16,** 607–617 (1980).
4. I. Suliciu, Some energetic properties of smooth solutions in rate-type viscoelasticity. *Int. J. Non-linear Mech.* **19,** 525–544 (1984).
5. D. R. Owen, *A First Course in the Mathematical Foundations of Thermodynamics.* Springer, Berlin (1984).
6. J. F. Bell, The experimental foundations of solid mechanics. In *Handbuch der Physik,* VI a/1. Springer, Berlin (1973).

7. V. V. Sokolovskii, The propagation of elastic–viscoplastic waves in bars. *Dokl. Akad. Nauk SSSR* **60**, 775–778 (1948); *Prikl. Mat. Mech.* **12**, 261–280 (1948) (in Russian).
8. L. E. Malvern, The propagation of longitudinal waves of plastic deformation in a bar of a material exhibiting a strain rate effect. *J. Appl. Mech.* **18**, 203–208 (1951). Plastic wave propagation in a bar of material exhibiting a strain rate effect. *Q. Appl. Math.* **8**, 405–411 (1951).
9. N. Cristescu, On the propagation of elastic–plastic waves in metallic rods. *Bull. Acad. Polon. Sci.* **11**, 129–133 (1963).
10. S. K. Godunov, *Equations of Mathematical Physics*. Nauka, Moscow (1979) (in Russian).